Rozhodněte, jestli mají limitu. Pokud mají, určete ji, jinak dokažte neexistenci.

a) $\lim_{n\to\infty} (-1)^{n!}$ $\quad (-1)^{n!} = \left(\left((-1)^2\right)\right)^{\frac{n!}{2}} = (1)^{\frac{n!}{2}} = 1$ $\quad \lim_{n\to\infty} (-1)^{n!} = \underline{\underline{1}}$

b) $\lim_{n\to\infty} \log(\sqrt{n}) = \lim_{n\to\infty} \frac{1}{2}\log n = \frac{1}{2}\cdot\lim_{n\to\infty} \log n = \frac{1}{2}\cdot\infty = \underline{\underline{+\infty}}$

c) $\lim_{n\to\infty} \frac{2^n+3^n+5^n}{2^{n+1}+3^{n+1}+5^{n+1}} = \frac{\cancel{5^{n+1}}\left(5^{-1}\left(\frac{2}{5}\right)^n + 5^{-1}\cdot\left(\frac{3}{5}\right)^n + 5^{-1}\right)}{\cancel{5^{n+1}}\left(\left(\frac{2}{5}\right)^{n+1} + \left(\frac{3}{5}\right)^{n+1} + 1^{n+1}\right)} \Rightarrow \frac{\frac{1}{5}\cdot 0 + \frac{1}{5}\cdot 0 + \frac{1}{5}}{0+0+1} = \underline{\underline{\frac{1}{5}}}$

$$\lim_{n\to\infty} \frac{2^n+3^n+5^n}{2^{n+1}+3^{n+1}+5^{n+1}} = \underline{\underline{\frac{1}{5}}}$$

d) $\lim_{n\to\infty} \frac{3n^2+2n+1}{n^2+2n+3} = \frac{\frac{3n^2}{n^2}+\frac{2n}{n^2}+\frac{1}{n^2}}{\frac{n^2}{n^2}+\frac{2n}{n^2}+\frac{3}{n^2}} = \frac{3+\frac{2}{n}+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{3}{n^2}} = \underline{\underline{3}}$

e) $\lim_{n\to\infty} \frac{2^n}{n!}$

Nechť $n > 2$: $\frac{2^n}{n!} = \frac{2}{1}\cdot\frac{2}{2}\cdot\frac{2}{3}\cdot\ldots\cdot\frac{2}{n}$ → bude násobeno primejmenším 2 a 1.

Tedy:

faktoriál ani druhá mocnina není záporná

$0 \leq \lim_{n\to\infty} \frac{2^n}{n!} \leq \lim_{n\to\infty} \frac{4}{n}$

$0 \leq \lim_{n\to\infty} \frac{2^n}{n!} \leq 0$ (věta o dvou policajtech) $\Rightarrow \lim_{n\to\infty} \frac{2^n}{n!} = \underline{\underline{0}}$

f) $\lim_{n\to\infty} \frac{\lfloor\sqrt{n}\rfloor}{\sqrt{n}}$ $\qquad \frac{\sqrt{n}-1}{\sqrt{n}} \leq \frac{\lfloor\sqrt{n}\rfloor}{\sqrt{n}} \leq \frac{\sqrt{n}}{\sqrt{n}}$

$$\underline{1} = \frac{1-\frac{1}{\sqrt{n}}}{1} = \frac{\sqrt{n}-1}{\sqrt{n}} \leq \frac{\lfloor\sqrt{n}\rfloor}{\sqrt{n}} \leq \frac{\sqrt{n}}{\sqrt{n}} = \underline{1}$$

$\lim_{n\to\infty} \frac{\lfloor\sqrt{n}\rfloor}{\sqrt{n}} = \underline{\underline{1}}$ (věta o dvou policajtech)

g) $\lim\limits_{n\to\infty} \sqrt{n}\cdot\left(\sqrt{n+1}-\sqrt{n-1}\right)$

$$\left(\sqrt{n^2+n}-\sqrt{n^2-n}\right)\cdot\frac{\sqrt{n^2+n}+\sqrt{n^2-n}}{\sqrt{n^2+n}+\sqrt{n^2-n}} = \frac{n^2+n-n^2+n}{\sqrt{n^2+n}+\sqrt{n^2-n}} = \frac{2n}{\sqrt{n^2\cdot\left(1+\frac{1}{n}\right)}+\sqrt{n^2\cdot\left(1-\frac{1}{n}\right)}} =$$

$$\frac{2n}{n\cdot\sqrt{1+\frac{1}{n}}+n\cdot\sqrt{1-\frac{1}{n}}} = \frac{2}{\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}} \Rightarrow \frac{2}{1+0+1+0} = \frac{2}{2} = 1$$

$$\Rightarrow \lim_{n\to\infty}\sqrt{n}\cdot\left(\sqrt{n+1}-\sqrt{n-1}\right) = \underline{\underline{1}}$$

h) $\lim\limits_{n\to\infty} \sqrt[n]{n^2+1}$

$$\sqrt[n]{n^2\cdot\left(1+\frac{1}{n^2}\right)} = n^{\frac{2}{n}}\cdot\sqrt[n]{1+\frac{1}{n^2}} \Rightarrow n^0\cdot\sqrt[n]{1+0} = 1\cdot 1 = \underline{\underline{1}}$$

$$\lim_{n\to\infty}\sqrt[n]{n^2+1} = \underline{\underline{1}}$$

Spočtěte limitu posloupnosti $(a_n)_{n=1}^{\infty}$ zadané následovně:

$$a_1 = 1, a_{n+1} = \frac{a_n^2}{4} + 1 \text{ pro } n = 1, 2, \dots .$$

$$\lim_{n\to\infty} a_n = A = \lim_{n\to\infty} a_{n+1} = \frac{A^2}{4}+1 \quad = \quad A = \frac{A^2+4}{4} \qquad \frac{A^2+4-4A}{4} = 0$$

$$A^2-4A+4 = 0$$

$$(A-2)\cdot(A-2)$$

$$A = 2$$

Důkaz existence limity:

a) je rostoucí

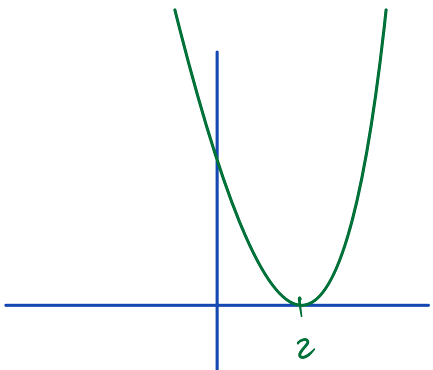
b) je shora omezená

(a) $a_n < a_{n+1}$

$$a_n < \frac{a_n^2}{4}+1$$

$$0 < \frac{a_n^2-4a_n+4}{4}$$

$$0 < a_n^2-4a_n+4$$




teda platí $\underline{\mathbb{R}\setminus\{2\}}$

(b) $\forall n: a_n < A = 2$

$n = 1$: $1 < 2$ ✓

1 : 1
2 : 1,25
3 : 1,39
4 : 1,48
5 : 1,55
6 : 1,60
7 : 1,64
8 : 1,67
9 : 1,69
10 : 1,72

$n \Rightarrow n+1$

$$a_{n+1} = \frac{a_n^2}{4} + 1 < \frac{A^2}{4} + 1 = 2$$

$$\frac{a_n^2}{4} + 1 < 2 \qquad \frac{a_n^2 - 4}{4} < 0 \qquad \frac{a_n^2 - 4}{4} < 0$$

$$\Rightarrow a_n^2 < 4 \quad \checkmark$$

Podle IP má $a_n$ limitu 2, tedy tato nerovnost platí. Zároveň nejmenší prvek $a_1 = 1 \in (-2, 2)$

$\boxed{X}$