a) $\int \frac{1}{x+\alpha}\,dx$ $t = x+\alpha$, $dt = dx$ $= \int \frac{1}{t}\,dt = \ln|t| = \ln|x+\alpha| + c$

b) $\int \frac{1}{(x+\alpha)^2}\,dx$ $t = (x+\alpha)$, $dt = 2x\,dx$ $= \int \frac{1}{t^2}\,dt = -\frac{1}{t} = -\frac{1}{x+\alpha} + c$

c) $\int \frac{1}{x^2+1}\,dx = \operatorname{arctg} x + c$

d) $\int \frac{2}{2}\frac{x}{x^2+1}\,dx$ $t = x^2+1$, $dt = 2x\,dx$ $= \frac{1}{2}\int \frac{1}{t}\,dt = \frac{1}{2}\ln|t| = \frac{1}{2}\ln(x^2+1) + c$!

dt

– tzn. že to rozšířím tak, abych měl ve finále jen $\frac{1}{t}$ a dt

Spočítejte:

Rozklad na parciální zlomky

a) $\int \frac{x^2+1}{x^2-1}\,dx = \int \left(1 + \frac{1}{x+1} - \frac{1}{x-1}\right)dx = x + \ln|x+1| + \ln|x-1| + c$

b) $\int \frac{x-2}{(x-1)^2}\,dx = \int \left(\frac{1}{x-1} - \frac{1}{(x-1)^2}\right)dx = \ln|x-1| + \frac{1}{x-1} + c$

c) $\int \frac{3}{x^2+2x+4}\,dx = \int$

$x^2+2x+4 = \text{„něco"}^2 + 1$ → arctg měl konstantu jen + 1

$x^2+2x+4 = (x+1)^2 + 3$

$= 3\cdot\left(\left(\frac{(x+1)}{\sqrt{3}}\right)^2 + 1\right)$

$t = \frac{(x+1)}{\sqrt{3}}$

$dt = \frac{1}{\sqrt{3}}\,dx$

→ $\int \frac{3}{3\cdot\left(\left(\frac{(x+1)}{\sqrt{3}}\right)^2+1\right)} = \frac{1}{\left(\left(\frac{(x+1)^2}{\sqrt{3}}\right)+1\right)}\frac{\sqrt{3}}{\sqrt{3}}dx = \frac{1}{\sqrt{3}}\int \frac{1}{t^2+1}\,dt =$

$= \sqrt{3}\,\operatorname{arctg} t = \sqrt{3}\,\operatorname{arctg}\left(\frac{x+1}{\sqrt{3}}\right) + c$

Rozklad na parciální zlomek:

$\frac{x^2+1}{x^2-1} = \frac{(x^2-1)+2}{(x^2-1)} = 1 + \frac{2}{(x^2-1)} + 0x =$

$1 + \frac{2}{(x-1)\cdot(x+1)}$

porovnat

$= 1 + \frac{\alpha}{x-1} + \frac{\beta}{x+1} = 1 + \frac{\alpha(x+1)+\beta(x-1)}{(x-1)\cdot(x+1)}$

$\alpha - \beta = 2$ $\alpha = 1$

$\alpha + \beta = 0$ $\beta = -1$

$\frac{x-2}{(x-1)^2} = \frac{\alpha}{(x-1)^1} + \frac{\beta}{(x-1)^2}$

$\frac{\alpha(x-1)+\beta}{(x-1)^2}$

$\alpha x - \alpha + \beta = x + 1$

$\alpha = 1$ $\beta = -1$

$-1 + \beta = -2$

Rozložte na parciální zlomky:

a) $$\frac{7x+2}{x^2+x-2} = \frac{7x+2}{(x+2)\cdot(x-1)} = \frac{\alpha}{(x+2)} + \frac{\beta}{(x-1)} \qquad \frac{\alpha x-\alpha+\beta x+2\beta}{(x^2+x-2)}$$

$$7x = \alpha x + \beta x$$
$$2 = -\alpha + 2\beta \qquad \alpha = 2\beta - 2$$
$$7x = 2\beta x - 2x + \beta x \qquad 2 = -\alpha + 6$$
$$9x = 3\beta x$$
$$\beta = 3 \qquad \alpha = 8$$

$$= \frac{8}{(x+2)} + \frac{3}{(x-1)}$$

Nebo

$$\frac{7x+2}{(x-1)\cdot(x+2)} = \frac{\alpha}{(x-1)} + \frac{\beta}{(x+2)}$$

$$x=1 \qquad \frac{7x+2}{(x+2)} = \alpha + \frac{\beta}{(x+2)}\cdot(x-1)$$

Můžu tedy takhle dosazovat částečným zkrácením

$$3 = \alpha + 0$$

Musím mít ale jednonásobné kořeny, jinak to nefunguje.

Integrujte!

a) $\ln^2 x \, dx$

b) $\operatorname{arctg} x \, dx$

$f = (\ln x)^2 \quad f' = \frac{2\cdot \ln x}{x}$

$g = x \quad g' = 1$

a) $$\int \ln^2 x \, dx = \int \ln^2 x \cdot 1 \, dx$$

$f = \ln x \quad f' = \frac{1}{x}$

$g = x \quad g' = 1$

$$= \int x\cdot \ln^2 x - 2\cdot \int \ln x \cdot 1 \, dx$$

$$= x\cdot \ln^2 x - 2\cdot\left(x \ln x - \int 1 \, dx\right)$$

$$= x\cdot \ln^2 x - 2x\ln x - 2x + c$$

$f = \operatorname{arctg} x \quad f' = \frac{1}{x^2+1}$

$g = x \quad g' = 1$

b) $$\int \operatorname{arctg} x \cdot 1 \, dx = x\cdot \operatorname{arctg} x - \int \frac{x}{x^2+1} \, dx =$$

$t = x^2+1$

$dt = 2x \, dx$

$$= x\cdot \operatorname{arctg} x - \frac{1}{2}\int \frac{1}{t} \, dt = x\cdot \operatorname{arctg} x - \frac{1}{2}\ln|t| = x\cdot \operatorname{arctg} x - \frac{1}{2}\ln(x^2+1) + c$$

Integrujte!

$$\int \sqrt{1-x^2}\,dx$$

$x \in (-1,1)$

substituce: $x = \sin t \Rightarrow t = \arcsin x$

$dx = \cos t\, dt$

nemusí tu být absolutní hodnota díky intervalu

$$= \int \sqrt{1-\sin^2 t}\cdot \cos t\, dt = \int \sqrt{\cos^2 t}\cdot \cos t \cdot dt = \int |\cos t| \cdot \cos t \cdot dt$$

$$= \int \cos^2 t\, dt = \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right) = \frac{1}{2}\left(t + \sin t \cos t\right) = \frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right) + c$$

$$\int \frac{1}{\sin x}\,dx$$

$x = 2t$

$dx = 2\,dt$

$\sin 2x = \sin x \cos x \cdot 2$

$$\int \frac{1}{\sin 2t}\cdot 2dt = \int \frac{1}{\cancel{2}\sin x \cos x}\cdot \cancel{2}dt = \frac{\sin^2 t + \cos^2 t}{\sin t \cdot \cos t}\cdot dt$$

$$= \int \frac{\sin t}{\cos t} + \frac{\cos t}{\sin t}\, dt = \int -\frac{1}{u}\,du + \int \frac{1}{v}\,dv = \ln|\sin t| - \ln|\cos t| = \ln\left|\frac{\sin t}{\cos t}\right| = \ln\left|\operatorname{tg}\frac{x}{2}\right|$$

$u = \cos t$ $\quad v = \sin t$

$du = -\sin t\, dt$ $\quad dv = \cos t\, dt$