Domácí úlohy — sada č. 8

Jméno a příjmení: Mikoláš Fromm, McCoolAus ....

Potřebný čas: MOC!

$$\langle x,y\rangle = \sum_{i=1}^{4} x_i y_i$$

1. V prostoru $\mathbb{R}^4$ se standardním skalárním součinem nalezněte ortonormální bázi řádkového prostoru následující matice

$$\begin{pmatrix} -4 & 1 & -2 & 2 \\ -6 & 4 & -8 & 3 \\ 8 & 3 & -6 & -4 \end{pmatrix} \sim \begin{pmatrix} -4 & 1 & -2 & 2 \\ 0 & 5 & -10 & 0 \\ -2 & 3 & -6 & 1 \end{pmatrix} \sim \begin{pmatrix} -4 & 1 & -2 & 2 \\ 0 & 5 & -10 & 0 \\ 0 & \frac{5}{2} & -5 & 0 \end{pmatrix} \sim \begin{pmatrix} -4 & 1 & -2 & 2 \\ 0 & 5 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \to u_1^T \\ \to u_2^T \\ \\ \end{matrix}$$

Poté získanou bázi doplňte na ortonormální bázi celého prostoru $\mathbb{R}^4$.

potřeba doplnit bázi $R(A)$, např.:

$u_3^T = (0,0,1,0)^T$

$u_4^T = (0,0,0,1)^T$

$i=1$:

$w_1 = (-4,1,-2,2)^T$

$v_1 = \frac{1}{\|w_1\|}\cdot w_1 = \frac{1}{\sqrt{16+1+4+4}}\cdot(-4,1,-2,2)^T = \left(-\frac{4}{5},\frac{1}{5},-\frac{2}{5},\frac{2}{5}\right)^T$

$i=2$:

$\langle u_2|v_1\rangle = \langle(0,5,-10,0)^T|(-\frac{4}{5},\frac{1}{5},-\frac{2}{5},\frac{2}{5})^T\rangle = 0+1+4+0$

$w_2 = (0,5,-10,0)^T - \sum_{i=1}^{1}\langle u_2|v_1\rangle v_1 = (0,5,-10,0) - 5\cdot\left(-\frac{4}{5},\frac{1}{5},-\frac{2}{5},\frac{2}{5}\right) = (4,4,-8,-2)^T$

$v_2 = \frac{1}{\|w_2\|}\cdot w_2 = \frac{1}{10}\cdot(4,4,-8,-2)^T = \left(\frac{2}{5},\frac{2}{5},-\frac{4}{5},-\frac{1}{5}\right)^T$

$\sqrt{16+16+64+4} = \sqrt{100}$

Ortonormální báze $R(A)$

$i=3$:

$\langle(0,0,1,0)^T|(-\frac{4}{5},\frac{1}{5},-\frac{2}{5},\frac{2}{5})^T\rangle = -\frac{2}{5}$

$w_3 = (0,0,1,0)^T - \langle u_3|v_1\rangle v_1 - \langle u_3|v_2\rangle v_2 = (0,0,1,0)^T - \left(\frac{8}{25},-\frac{2}{25},\frac{4}{25},-\frac{4}{25}\right)^T - \left(-\frac{8}{25},-\frac{8}{25},\frac{16}{25},\frac{4}{25}\right)^T = \left(0,\frac{10}{25},\frac{5}{25},0\right)^T$

$\langle(0,0,1,0)^T|(\frac{2}{5},\frac{2}{5},-\frac{4}{5},-\frac{1}{5})^T\rangle = -\frac{4}{5}$

$v_3 = \frac{1}{\|w_3\|}\cdot w_3 = \frac{1}{\frac{1}{\sqrt5}}\cdot\left(0,\frac{2}{5},\frac{1}{5},0\right)^T = \sqrt5\cdot\left(0,\frac{2}{5},\frac{1}{5},0\right)^T = \left(0,\frac{2\sqrt5}{5},\frac{1\sqrt5}{5},0\right)^T$

$\sqrt{0+\frac{4}{25}+\frac{1}{25}+0} = \sqrt{\frac{5}{25}} = \frac{\sqrt5}{5} = \frac{1}{\sqrt5}$

$i=4$:

$\langle(0,0,0,1)^T|(\cdot,\cdot,\cdot,\frac{2}{5})^T\rangle = \frac{2}{5}$ $\quad$ $\langle(0,0,0,1)^T|(\cdot,\cdot,\cdot,0)^T\rangle = 0$

$w_4 = (0,0,0,1)^T - \langle u_4|v_1\rangle v_1 - \langle u_4|v_2\rangle v_2 - \langle u_4|v_3\rangle v_3 = (0,0,0,1)^T - \left(-\frac{8}{25},\frac{2}{25},-\frac{4}{25},\frac{4}{25}\right)^T - \left(-\frac{2}{25},-\frac{2}{25},+\frac{4}{25},+\frac{1}{25}\right)^T - 0 = \left(\frac{10}{25},0,0,\frac{20}{25}\right)^T$

$\langle(0,0,0,1)^T|(\cdot,\cdot,\cdot,-\frac{1}{5})^T\rangle = -\frac{1}{5}$

$v_4 = \frac{1}{\|w_4\|}\cdot w_4 = \frac{\sqrt5}{2}\cdot\left(\frac{2}{5},0,0,\frac{4}{5}\right)^T = \left(\frac{2\sqrt5}{10},0,0,\frac{4\sqrt5}{10}\right)^T = \left(\frac{\sqrt5}{5},0,0,\frac{2\sqrt5}{5}\right)^T = \left(\frac{1}{\sqrt5},0,0,\frac{2}{\sqrt5}\right)^T$

$\sqrt{\frac{4}{25}+0+0+\frac{16}{25}} = \sqrt{\frac{20}{25}} = \frac{\sqrt{20}}{5} = \frac{2\sqrt5}{5} = \frac{2}{\sqrt5}$

Doplnění báze do celého prostoru $\mathbb{R}^4$

pokračování na 2. straně

2. Najděte $\mathbf{x}'$ takové, že minimalizuje $||\mathbf{A}\mathbf{x}' - \mathbf{b}||$, kde

Všechny vektory jsou opět kolmé!
Stačí tedy udělat projekci na základě jejich normalizované formy

$$\begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \mathbf{A} = \begin{pmatrix} 3 & 2 & -1 & 1 & 3 \\ 3 & 1 & 3 & -1 & -2 \\ 2 & -3 & -1 & -1 & -1 \\ 1 & -1 & -2 & -3 & 1 \\ 1 & -3 & 1 & 3 & 1 \\ 1 & 1 & -3 & 2 & -3 \end{pmatrix} \end{array}, \qquad \mathbf{b} = (26, 5, 34, -18, -30, -13)^T$$

Určete také hodnotu $||\mathbf{A}\mathbf{x}' - \mathbf{b}||$. (Jde o normu vzhledem ke standardnímu skalárnímu součinu).

$$||u_1|| = \sqrt{9+9+4+1+1+1} = \sqrt{25} = 5 \qquad ||u_2|| = \sqrt{4+1+9+1+9+1} = 5$$

$$v_1 = \frac{1}{5}(3,3,2,1,1,1)^T = \left(\frac{3}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)^T \qquad v_2 = \frac{1}{5}\cdot(2,1,-3,-1,-3,1)^T = \left(\frac{2}{5}, \frac{1}{5}, -\frac{3}{5}, -\frac{1}{5}, -\frac{3}{5}, \frac{1}{5}\right)^T$$

$$||u_3|| = \sqrt{1+9+1+4+1+9} = 5 \qquad ||u_4|| = \sqrt{1+1+1+9+9+4} = 5$$

$$v_3 = \frac{1}{5}(-1,3,-1,-2,1,-3)^T = \left(-\frac{1}{5}, \frac{3}{5}, -\frac{1}{5}, -\frac{2}{5}, \frac{1}{5}, -\frac{3}{5}\right)^T \qquad v_4 = \frac{1}{5}(1,-1,-1,-3,3,2)^T = \left(\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{3}{5}, \frac{3}{5}, \frac{2}{5}\right)^T$$

$$||u_5|| = \sqrt{9+4+1+1+1+9} = 5$$

$$v_5 = \frac{1}{5}\cdot(3,-2,-1,1,1,-3)^T = \left(\frac{3}{5}, -\frac{2}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, -\frac{3}{5}\right)^T$$

x' odpovídá Fourierovým koeficientům!

$$x' = (20, 10, 0, -15, 5)^T = (4, 2, 0, -3, 1)^T$$

$$P_2(b) = \overbrace{\langle b|v_1\rangle}^{20} v_1 + \underbrace{\langle b|v_2\rangle}_{10} v_2 + \overbrace{\langle b|v_3\rangle}^{0} v_3 + \underbrace{\langle b|v_4\rangle}_{-15} v_4 + \overbrace{\langle b|v_5\rangle}^{5} v_5$$

$$b' = (12+4+0-3+3,\ 12+2+0+3-2,\ 8-6+0+3-1,\ 4-2+0+9+1,\ 4-6+0-9+1,\ 4+2+0-6-3)$$

$$b' = (16, 15, 4, 12, -10, -3)^T$$

$$||Ax' - b|| = ||(16, 15, 4, 12, -10, -3)^T - (26, 5, 34, -18, -30, -13)^T|| = \underline{50}$$