1)

$u \perp v$ Určete $\|u+v\|$ a $\|u-v\|$

$\|u\| = 12$

$\|v\| = 5$

$\langle u, v \rangle = 0$

$$\|u+v\| = \sqrt{\langle u+v | u+v \rangle} = \sqrt{\langle u+v | u \rangle + \langle u+v | v \rangle} = \sqrt{\overline{\langle u | u+v \rangle} + \overline{\langle v | u+v \rangle}}$$

$$= \sqrt{\overline{\langle u|u\rangle} + \underbrace{\overline{\langle u|v\rangle}}_{0} + \underbrace{\overline{\langle v|u\rangle}}_{0} + \overline{\langle v|v\rangle}} = \sqrt{\overline{\langle u|u\rangle} + \overline{\langle v|v\rangle}} =$$

$$= \sqrt{25 + 144} = \sqrt{169} = 13$$

2)

Gram-Schmidt algoritmus:

for i in n:

$$w_i = u_i - \sum_{j=1}^{n-1} \langle u_i | v_j \rangle v_j$$

$$v_i = \frac{1}{\|w_i\|} \cdot w_i$$

↗ vstup je $u_1 \_ u_n$ je báze

↘ výstup je $v_1 - v_n$ je ortonor. báze

Finální orto vektory

a)

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 4 & 1 & 4 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{matrix} \dots u_1^T \\ \dots u_2^T \\ \dots u_3^T \end{matrix}$$

$i=1$: $w_1 = u_1 - \sum_{j=1}^{0} = (1,1,1,1)^T$

$$v_1 = \frac{1}{\|w_1\|} \cdot w_1 = \frac{1}{\sqrt{4}} \cdot (1,1,1,1)^T = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^T$$

$i=2$: $w_2 = u_2 - \sum_{j=1}^{1} \langle u_2 | v_1 \rangle v_1 = (4,1,4,1)^T - 5 \cdot \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^T$

$$= \left(\frac{3}{2}, -\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}\right)^T$$

$$\langle u_2 | v_1 \rangle = \langle (4,1,4,1)^T | \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)^T \rangle = 5$$

$$\|w_2\| = \sqrt{\frac{36}{4}} = 3$$

$$v_2 = \frac{1}{\|w_2\|} \cdot w_2 = \frac{1}{3} \cdot \left(\frac{3}{2}, -\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}\right)^T = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)^T$$

$i=3$: $w_3 = u_3 - \langle u_3 | v_1 \rangle v_1 - \langle u_3 | v_2 \rangle v_2 = (1,2,3,4)^T - 5 \cdot \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$$+ 1 \cdot \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)^T$$

$$= (-1, -1, 1, 1)^T$$

$$\langle u_3 | v_1 \rangle = \langle (1,2,3,4)^T | \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)^T \rangle = 5$$

$$\langle u_3 | v_2 \rangle = \langle (1,2,3,4)^T | \left(\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\right)^T \rangle = -1$$

$$\|w_3\| = \sqrt{4} = 2$$

$$v_3 = \frac{1}{\|w_3\|} \cdot w_3 = \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^T$$

3) Rozšiřte bázi z $\mathbb{R}^3$ na $\mathbb{R}^4$, kde báze $:= \left\{ \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)^T, \left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)^T, \left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)^T \right\}$

– takže chceme u_4 takový, že $u_4 \notin \mathcal{R}(A)$ -> tedy že je LNZ na vektory báze

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 4 & 1 & 4 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 0 & -3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} \underline{1} & 1 & 1 & 1 \\ 0 & \underline{1} & 0 & 1 \\ 0 & 0 & \underline{1} & 1 \end{pmatrix}$$

-> lze zvolit $u_4 = (0,0,0,1)^T$

$$w_4 = u_4 - \langle u_4|v_1\rangle v_1 - \langle u_4|v_2\rangle v_2 - \langle u_4|v_3\rangle v_3$$

$$(0,0,0,1)^T - \frac{1}{2}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) - \left(-\frac{1}{2}\right)\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right) - \frac{1}{2}\left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)^T = \left(\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4}\right)^T$$

$$v_4 = \frac{1}{2} \cdot w_4 = \frac{1}{2} \cdot w_4 = \left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)^T$$

4) Určete ortogonální projekci p vektoru $a = (2,2,1,5)^T$ do $\mathcal{R}(A)$ a souřadnice $[p]_Z$

$$P_{\mathcal{R}(A)}(a) = \langle a|v_1\rangle v_1 + \langle a|v_2\rangle v_2 + \langle a|v_3\rangle v_3 = 5\cdot\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)^T - 2\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)^T + 1\left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)^T =$$

Toto je ta projekce $\Big\} \longrightarrow$ $= (1,4,2,5)^T$

6)
$$A = \begin{array}{c} \begin{matrix} u_1 & u_2 & u_3 \end{matrix} \\ \begin{pmatrix} 2 & 1 & 0 \\ 4 & 2 & 0 \\ 2 & -4 & -1 \\ 1 & -2 & 2 \end{pmatrix} \end{array}$$

-> všechny jsou kolmé
– aby to byly vektory ortonormální báze, musí mít vlastní normu rovnou 1.

$b = (10,5,13,9)^T$

$\|u_1\| = \sqrt{4+16+4+1} = 5$ $\quad \|u_2\| = \sqrt{1+4+16+4} = 5$ $\quad \|u_3\| = \sqrt{1+4} = \sqrt{5}$

$z_1 = \left(\frac{2}{5},\frac{4}{5},\frac{2}{5},\frac{1}{5}\right)^T$ $\quad z_2 = \left(\frac{1}{5},\frac{2}{5},-\frac{4}{5},-\frac{2}{5}\right)^T$ $\quad z_3 = \left(0,0,-\frac{\sqrt{5}}{5},\frac{2\sqrt{5}}{5}\right)^T$

$$b' = P_Z(b) = \overset{4+4+7}{\langle b|z_1\rangle} z_1 + \underset{2+2-14}{\langle b|z_2\rangle} z_2 + \overset{\sqrt{5}}{\langle b|z_3\rangle} z_3 = 15\left(\frac{2}{5},\frac{4}{5},\frac{2}{5},\frac{1}{5}\right)^T - 10\left(\frac{1}{5},\frac{2}{5},-\frac{4}{5},-\frac{2}{5}\right)^T + \sqrt{5}\left(0,0,-\frac{\sqrt{5}}{5},\frac{2\sqrt{5}}{5}\right)^T$$

$b' = (4,8,13,9)^T$ $\qquad x' = (15,-10,\sqrt{5})^T$